Solvay Brussels School Economics & Management

ULB

Financial Risk Management and Governance

Volatilities

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Definition

- The volatility of a variable is the standard deviation of its return with the return being expressed with continuous compounding
- The variance rate is the square of volatility
- Implied volatilities are the volatilities implied from option prices
- Normally days when markets are closed are ignored in volatility calculations (252 days)

$$\sigma\sqrt{T} = \text{standard deviation of } \ln\left(S(T)/S(0)\right)$$

$$\sqrt{\underbrace{\frac{1}{(n-1)}\sum_{1}^{T}}_{\text{the same weight.}}(\mu_i-\bar{\mu})^2}$$

- When T is small
  - » The continuously compounded return of a market variable is close to the percentage change
  - » $\sigma\sqrt{T} \approx$ std. deviation of % change in market variable

Some thoughts on implied volatilities...

(Pricing)

→ implied volat. (most recent)

↳ deduced from option prices (ATM)

combines

→ a current estimation of the market volatility

→ + the "desire" to be protected = the "how much expensive" is the insurance = risk aversion.

Trader

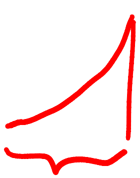
(RM)

→ historical volat.

(a) we use a period time window & we may "slide" it to recalculate the volat.

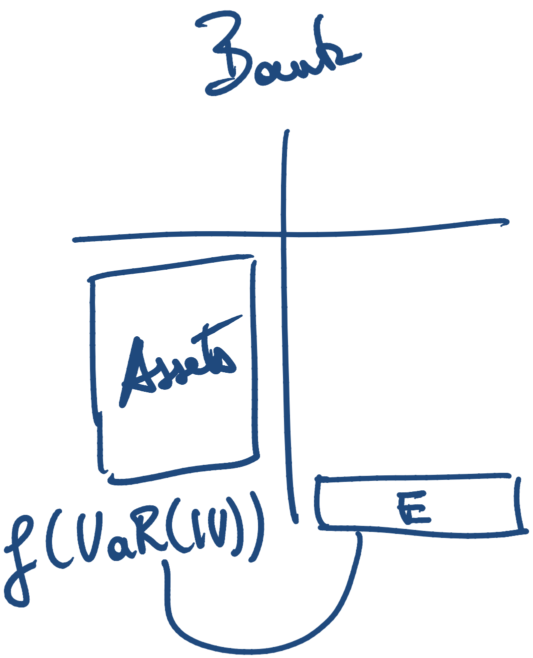
(b) we would like to make the historical volat more "weighted" to recent observations

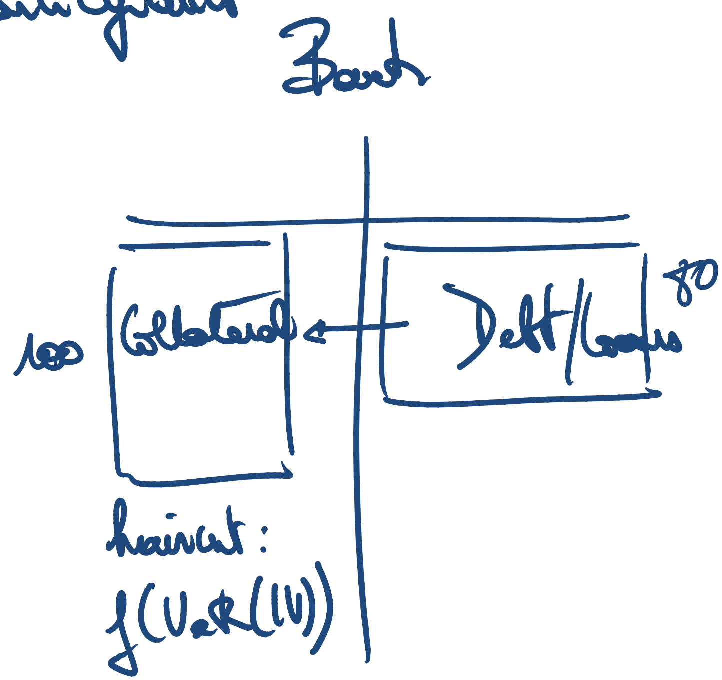
Risk management



Σ weights = 1

- in RM, we use volatilities to
  - measure risk
  - to compute the equity buffer of banks
  - for VaR for the value of collateral.
- in IV
  - ▷ environment is more risky.
  - ▷ riskophobia of market participants





The VIX index

Implied Volat. Index

From historical data...

- We have
  - *n+1* price observations
  - Returns: i=1, 2..., n

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

  - The estimate of the standard deviation of u_i is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

or

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n}u_i\right)^2}$$

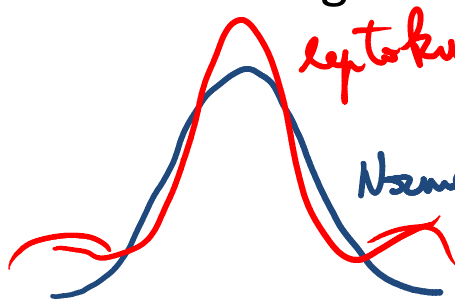
  - *s* is an estimate of $\sigma\sqrt{T}$
  - Therefore: $\hat{\sigma} = \frac{s}{\sqrt{T}}$ with standard error $\approx \hat{\sigma}/\sqrt{2n}$

Are Daily Changes in Exchange Rates Normally Distributed?

- Knowing this...

	Real World (%)	Normal Model (%)
>1 SD	25.04	31.73
>2SD	5.27	4.55
>3SD	1.34	0.27
>4SD	0.29	0.01
>5SD	0.08	0.00
>6SD	0.03	0.00

- What should you do if you observe fat tails in the tails of a market variable while the market assumes log-normality of prices?
- Statistical jargon:
  - » Leptokurtic
  - » Platykurtic
  - » Mesokurtic



leptokurtosis

Normal = mesokurtic.

"jumps"

changes in volatility

The power law

- Formulation
  - For many variables, when *x* is large (*K* and α are constants)

$$\boxed{\Pr(v > x) = Kx^{-\alpha}}$$

- This seems to fit the behavior of the returns on many market variables better than the normal distribution

- The previous equation implies that

$$\boxed{\ln\left[\Pr(v > x)\right] = \ln K - \alpha \ln x}$$

- This law will be used when we will be talking about Extreme Value Theory and Operational Risk.

Standard Approach to Estimating Volatility

- Define σ_n as the volatility per day between day n-1 and day n, as estimated at end of day n-1
- Define S_i as the value of market variable at end of day i
- Define $u_i = \ln(S_i/S_{i-1})$

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \bar{u})^2$$

$$\bar{u} = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}$$

Simplifications Usually Made in Risk Management

- Define u_i as $(S_i\text{-}S_{i\text{-}1})/S_{i\text{-}1}$
- Assume that the mean value of u_i is zero
- Replace m-1 by m

This gives $$\boxed{\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2}$$

Weighting Scheme

- Instead of assigning equal weights to the observations we can set

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where

$$\sum_{i=1}^{m} \alpha_i = 1$$

function that weights the observation differently

ARCH(m) model

- In an ARCH(m) model we also assign some weight to the long-run variance rate, V_L:

$$\sigma_n^2 = \underbrace{\gamma V_L}_{} + \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

long-term trend

where

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$

ARCH(1) : $\sigma_n^2 = \gamma . V_L + (1-\gamma) . u_{n-1}^2$

EWMA

- In an exponentially weighted moving average model, the weights assigned to the u^2 decline exponentially as we move back through time

$\lambda . \sigma^2_{n-2} + (1-\lambda) . u^2_{n-2}$

- This leads to

$$\boxed{\begin{aligned}\sigma_n^2 &= \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2 \\ &= \underbrace{(1-\lambda)\sum_{i=1}^{m}\lambda^{i-1}u_{n-i}^2}_{\text{weighted sum of past returns}} + \underbrace{\lambda^m\sigma_{n-m}^2}_{\text{the residual } \sigma^2 \text{ far in the past.}}\end{aligned}}$$

decreases exponentially as you move backward

- Advantages
  - » Relatively little data needs to be stored
  - » We need only remember the current estimate of the variance rate and the most recent observation on the market variable
  - » Tracks volatility changes
  - » RiskMetrics uses λ = 0.94 for daily volatility forecasting

GARCH (1,1)

autoregressive conditional heteroskedasticity.

- In GARCH (1,1) we assign some weight to the long-run average variance rate

$$\boxed{\sigma_n^2 = \underbrace{\gamma V_L}_{\text{LT trend}} + \underbrace{\alpha u_{n-1}^2}_{\text{new returns}} + \underbrace{\beta \sigma_{n-1}^2}_{\text{auto-correlation to previous measures}}}$$

ML

Since weights must sum to 1

$$\gamma + \alpha + \beta = 1$$

- Setting $\omega = \gamma V$ the GARCH (1,1) model is $\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$ and $V_L = \dfrac{\omega}{1-\alpha-\beta}$

$$\frac{\omega}{1-\alpha-\beta} = V_L$$

- Example

$$\sigma_n^2 = \overbrace{0.000002}^{\omega} + \overbrace{0.13}^{\alpha}u_{n-1}^2 + \overbrace{0.86}^{\beta}\sigma_{n-1}^2$$

  » Suppose

  » The long-run variance rate is 0.0002 so that the long-run volatility per day is 1.4%

GARCH (1,1)

- Example 2
  - » Suppose that the current estimate of the volatility is 1.6% per day and the most recent percentage change in the market variable is 1%.
  - » The new variance rate is

$$0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023336$$

The new volatility is 1.53% per day

GARCH (p,q)

$$\sigma_n^2 = \omega + \sum_{i=1}^{p} \alpha_i u_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2$$

Others

- We can design GARCH models so that the weight given to u_i^2 depends on whether u_i is positive or negative
- We do not have to assume that the conditional distribution is normal

Variance targeting

- One way of implementing GARCH(1,1) that increases stability is by using variance targeting
- We set the long-run average volatility equal to the sample variance
- Only two other parameters then have to be estimated

Maximum likelihood

- In maximum likelihood methods we choose parameters that maximize the likelihood of the observations occurring
- Example 1
  - We observe that a certain event happens one time in ten trials. What is our estimate of the proportion of the time, *p*, that it happens?
  - The probability of the outcome is $p(1-p)^9$
  - We maximize this to obtain a maximum likelihood estimate: p=0.1 (differentiate with respect to *p* and set result equal to 0)
- Example 2
  - Estimate the variance of observations from a normal distribution with mean zero

Maximize: $$\prod_{i=1}^{m}\left[\frac{1}{\sqrt{2\pi v}}\exp\left(\frac{-u_i^2}{2v}\right)\right]$$

or: $$\sum_{i=1}^{m}\left[-\ln(v)-\frac{u_i^2}{v}\right]=-m\ln(v)-\sum_{i=1}^{m}\left[\frac{u_i^2}{v}\right]$$

This gives: $$v=\frac{1}{m}\sum_{i=1}^{m}u_i^2$$

Application to GARCH(1,1)

- We choose parameters that maximize $$\sum_{i=1}^{n}\left[-\ln(v_i)-\frac{u_i^2}{v_i}\right]$$